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# Charged singularities: the causality violation

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**Abstract.** We search for examples of particle trajectories which, approaching a naked singularity from infinity, make up for lost time before going back to infinity. In the Kerr–Newman metric we found a whole family of such trajectories, showing therefore that the causality violation is indeed a non-avoidable pathology.

## 1. Introduction

It is well known (Tippler 1976) that closed time-like paths may exist in the field of a naked singularity (NS). These paths imply the possibility that any two points of the space–time manifold can be connected by either a future or past directed time-like path. This is the case for the Kerr and Kerr–Newman NS solutions (Carter 1968).

Here closed time-like paths are found in a restricted region of space–time where the axial Killing vector becomes time-like. A time-like path entering this region was believed to travel into the past with respect to an observer at infinity.

In a previous work, however, it was shown that entering this region is not a sufficient condition for this peculiar phenomenon to take place; in fact, a more stringent condition has to be satisfied on the path itself, that is the gradient of the coordinate time has to vanish and become negative (Calvani *et al* 1978). Now, while the former (necessary) condition is satisfied at a given space–time point independently of the path, the sufficient condition (CV condition) does depend on the path so that one has to locate, on each path, where this condition is satisfied.

It was found that, on a particular class of time-like geodesics in the Kerr metric, the CV condition was never satisfied because these trajectories had a turning point just before. This suggested the possibility that no unbound path violated causality in the sense of never satisfying the sufficient condition. The first attempt to generalise our previous result was to consider the null geodesics (de Felice and Calvani 1979), and we found that some of them did violate causality although under restrictive conditions.

The objection remains that an arbitrary path (particle moving with an arbitrary acceleration) could always be thought as being able to violate causality (with respect to infinity).

One example of non-geodesic equations of motion which have been completely solved is given by those of a charged particle in the Kerr–Newman family of space–time solutions of Einstein's equations. Here the term which regulates the trajectory of the particle is its charge. Our aim is therefore to investigate whether a charged particle can be steered to the fulfilment of its CV condition.

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In this paper we explicitly find a family of accelerated trajectories which do fulfil their CV condition before meeting a turning point and going back to infinity. In § 2 we consider the Kerr–Newman space–time and define the necessary and sufficient conditions for causality violation with respect to infinity, and then we specialise them to the generalised radial trajectories (vortical motion, de Felice and Calvani (1972), Bicak and Stucklik (1976)). In § 3 we prove that particles oppositely charged to the source and moving on radial trajectories can violate causality. Throughout the paper geometrised units are used with $C = G = 1$.

## The Kerr–Newman space–time and the conditions for causality violation

The Kerr–Newman space–time solution is the charged generalisation of the Kerr solution, and reads in oblate spheroidal coordinates

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\sin^2\theta}{\Sigma}[(r^2+a^2)\,d\phi - a\,dt]^2 + \frac{\Delta}{\Sigma}dr^2 + \Sigma\, d\theta^2 \tag{1}$$

where

$$\Delta = r^2 + a^2 + Q^2 - 2Mr, \qquad \Sigma = r^2 + a^2\cos^2\theta. \tag{2}$$

Here $M$, $a$ and $Q$ are the mass, the specific angular momentum and the charge of the metric source.

The necessary condition for causality violation is given, as is well known, by

$$g_{\phi\phi} = (A/\Sigma)\sin^2\theta \leqslant 0 \tag{3}$$

where

$$A = (r^2+a^2)^2 - a^2\Delta\sin^2\theta. \tag{4}$$

Condition (3) identifies the region (hereafter called the $\phi$ region) where the axial Killing vector is no longer space-like; it implies

$$A \leqslant 0 \tag{5}$$

which leads to

$$\sin^2\theta \equiv \Phi^2 \geqslant (r^2+a^2)^2/a^2\Delta \equiv \Phi_\phi^2. \tag{6}$$

Hereafter we shall assume that the naked singularity condition holds:

$$a^2 + Q^2 > M^2 \tag{7}$$

which implies $\Delta > 0$ always.

The boundary of the $\phi$ region, where the axial Killing vector becomes null, is given by the function $\Phi_\phi^2$ in (6). It is obvious that $\Phi_\phi^2 > 0$ always, while $\Phi_\phi^2 = 1$ ($\theta = \pi/2$) where

$$Q^2 = (r/a^2)[r^3 + a^2(r+2M)] \equiv Q_1^2; \tag{8}$$

this function is shown in figure 1($b$). The loci of points where $Q_1^2$ in (8) attains a minimum value and where it is zero (besides $r = 0$) are given by the functions

$$a_1^2 = -\frac{2r^3}{r+M}, \qquad a_2^2 = -\frac{r^3}{r+2M}, \tag{9}$$

respectively; these are shown in figure 1($a$). With the help of figures 1($a$) and 1($b$) one

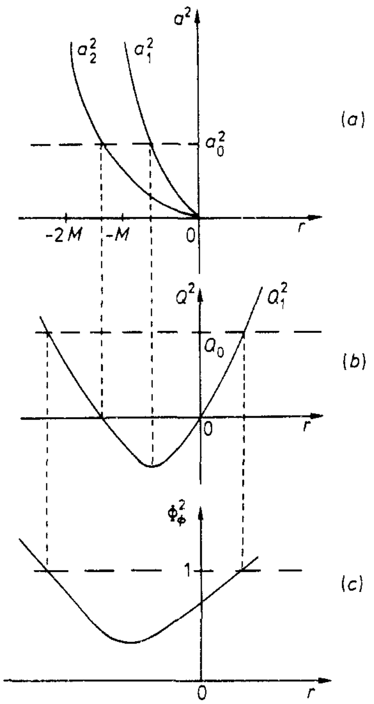


**Figure 1.** (*a*) The functions $a_1^2$ and $a_2^2$ are shown; they give respectively the minima and zeros of $Q_1^2$ for a chosen value of $a$, e.g. $a_0$. (*b*) The function $Q_1^2$ is shown; it gives the locus where $\Phi_\phi^2 = 1$ for a chosen value of $Q$, e.g. $Q_0$. (*c*) The function $\Phi_\phi^2$ in (6) is shown.

can immediately recognise the shape of the $\phi$ region, as shown in figure 1(*c*), for a given value of $Q$. One clearly sees that now, contrary to what happens in the Kerr metric, the $\phi$ region extends to positive $r$.

We have already pointed out that a general time-like path which enters the $\phi$ region is liable to violate causality with respect to infinity. However, this happens only if somewhere on the path

$$dt/d\lambda = 0 \tag{10}$$

where $\lambda$ is a parameter on the world line. This condition is consistent with the path being time-like if and only if $g_{\phi\phi} < 0$, so that the point where (10) holds is inside the $\phi$ region.

Because of the electromagnetic interaction, a test particle of mass $\mu$ and charge $e$ will not move along a geodesic; nevertheless the equations of motion have been solved, and read (Carter 1968)

$$\begin{aligned}
\Sigma\, dt/d\lambda &= -a(aE\sin^2\theta - l) + (r^2+a^2)P/\Delta, \\
\Sigma\, dr/d\lambda &= \pm\{P^2 - \Delta[\mu^2 r^2 + (l-aE)^2 + (L-l^2)]\}^{1/2}, \\
\Sigma\, d\theta/d\lambda &= \pm\{(L-l^2) - \cos^2\theta[a^2(\mu^2-E^2) + l^2/\sin^2\theta]\}^{1/2}, \\
\Sigma\, d\phi/d\lambda &= -(aE - l/\sin^2\theta) + aP/\Delta,
\end{aligned} \tag{11}$$

where $P = E(r^2+a^2) - al - eQr$. Here the parameters $E$ and $l$ are the total energy with respect to infinity and the axial angular momentum of the particle, while $L$ is related to the square of the total angular momentum (de Felice 1980).

From the first equation of (11), the condition $\mathrm{d}t/\mathrm{d}\lambda = 0$ identifies a well defined spatial region (hereafter called CV region) which depends on the test particle's parameters. Solving equation (10) with respect to $\sin^2\theta \equiv \Phi^2_{\mathrm{CV}}$, one has

$$\Phi^2_{\mathrm{CV}} = \Phi^2_{\phi} + \frac{Q^2 - 2Mr}{a\Delta}\left(\frac{l}{E}\right) - \frac{eQr(r^2+a^2)}{a^2\Delta E}. \tag{12}$$

Let us now point out that from (2) and (4), the condition $A \leqslant 0$ implies

$$2Mr - Q^2 < 0; \tag{13}$$

furthermore the requirement that the CV region is always inside the $\phi$ region implies

$$\frac{Q^2 - 2Mr}{a\Delta}\left(\frac{l}{E}\right) - \frac{eQr(r^2+a^2)}{a^2\Delta E} > 0. \tag{14}$$

We shall now consider a particular class of trajectories which can cross the $\phi$ region, that is the generalised radials; these are of the vortical type and are confined on the hyperboloids $\theta =$ constant with parameters given by

$$l = \epsilon a\sqrt{\Gamma}\sin^2\theta, \qquad \epsilon = \pm 1, \tag{15}$$

where $\Gamma = E^2 - \mu^2$. Let us assume that $\Gamma > 0$ (unbound motion) and call $\beta = eQ$, $G = [(\Gamma+1)/\Gamma]^{1/2}$; condition (12) now becomes

$$\Phi^2_{\mathrm{CV}} = \left(\Phi^2_{\phi} - \frac{\beta r(r^2+a^2)}{a^2\Delta(\Gamma+1)^{1/2}}\right)\left(1 - \epsilon\frac{Q^2-2Mr}{\Delta G}\right)^{-1}. \tag{16}$$

## 3. Causality violating trajectories

As we said in the Introduction, the most interesting situation arises when the test particle is endowed with a charge. In order to violate causality with respect to infinity, the test particle should enter the CV region, meet a turning point and then go back to infinity. We shall show that this happens for a family of trajectories; these were found by comparing the locus of the inversion points with the locus where the condition for causality violation, $\mathrm{d}t/\mathrm{d}\lambda \leqslant 0$, holds.

The value of $\Phi^2_{\mathrm{CV}}$, equation (12), at $r = 0$, does not depend on $\beta$ (and is always $<1$) so that, in order to satisfy the condition $\Phi^2_{\mathrm{CV}} > \Phi^2_{\phi}$, we must have $l > 0$, from (12) and (13). The axial angular momentum must therefore have a well defined sign; specialising to the generalised radial trajectories, one finds from (12) and (15) that

$$\Phi^2_{\mathrm{CV}} = \left(\frac{r^2+a^2}{a^2}\right)\frac{G(r^2+a^2) - \beta r(G^2-1)^{1/2}}{G(r^2+a^2) + (G-1)(Q^2-2Mr)}. \tag{17}$$

It is easy to prove that this function is monotonic increasing with $r$.

In a previous work (de Felice *et al* 1980) it was shown that the repulsive barriers (loci of inversion points) for charged particles moving on radial trajectories (with $l > 0$) are given by

$$\Phi^2_{\pm} = \frac{1}{a^2}\left\{G(r^2+a^2) + (G-1)\left[-\Delta - \beta r\left(\frac{G+1}{G-1}\right)^{1/2} \pm (\Delta\eta)^{1/2}\right]\right\} \tag{18}$$

where

$$\eta = Q^2 - 2Mr + 2\beta r\left(\frac{G+1}{G-1}\right)^{1/2}. \tag{19}$$

Let us first prove that we have always

$$\Phi^2_{\mathrm{CV}} > \Phi^2_-. \tag{20}$$

After some algebra, this inequality implies

$$\beta r(G^2-1)^{1/2} + (G-1)(Q^2-2Mr) + [(r^2+a^2)/\Delta + (G-1)](\Delta\eta)^{1/2} \geqslant 0. \tag{21}$$

Let us now consider the following two cases:

(a) $\beta > 0$

From condition (13) it follows that the inequality (21) is indeed satisfied.

(b) $\beta < 0$

The inequality (20) can be written as

$$(G-1)\eta - \beta r(G^2-1)^{1/2} \geqslant -[(r^2+a^2)/\Delta + (G-1)](\Delta\eta)^{1/2}; \tag{22}$$

here $\eta$ must be positive, so relation (22) is also satisfied; the condition (20) therefore holds, whatever is the sign of $\beta$.

Let us now compare $\Phi^2_{\mathrm{CV}}$ with $\Phi^2_+$. Let $\tilde{r}$ be the solution of $\eta = 0$, i.e.

$$\tilde{r} = Q^2 \bigg/ \left\{2\left[M - \beta\left(\frac{G+1}{G-1}\right)^{1/2}\right]\right\}; \tag{23}$$

we shall be interested in positive $\tilde{r}$ only. In de Felice *et al* (1980) it was shown that at $\tilde{r}$

$$\Phi^2_\pm(r=\tilde{r}) \geqslant 1 \qquad \text{when} \qquad Q^2 \geqslant -2\beta(G^2-1)^{1/2}\left[M - \beta\left(\frac{G+1}{G-1}\right)^{1/2}\right]. \tag{24}$$

Consider now the following two cases.

(c) $\beta > 0$

From (24) it follows that $\Phi^2_+(r=\tilde{r})$ is always $>1$, so that one has to compare $\Phi^2_{\mathrm{CV}}$ only with $\Phi^2_-$; but then (20) holds and there is no causality violation (see figure 3(*a*)).

(d) $\beta < 0$

Causality violation may now occur. After some algebra one can show that

$$\Phi^2_+(r=\tilde{r}) \leqslant 1 \qquad \text{when} \qquad \tilde{r} \leqslant [\beta^2(G^2-1)]^{1/2} \equiv \tilde{r}_0 \tag{25}$$

and

$$\Phi^2_{\mathrm{CV}}(r=\tilde{r}) \leqslant 1 \qquad \text{when} \qquad f(\tilde{r}) \equiv \tilde{r}^3 + \tilde{r}_0\tilde{r}^2/G + a^2\tilde{r} - \tilde{r}_0 a^2/G \leqslant 0. \tag{26}$$

The function $f(\tilde{r})$ is shown in figure 2: note that $f(\tilde{r}_0) > 0$ and $f(0) < 0$. For the values of the parameters $a$, $Q$, $M$ and $\beta$ which correspond to the shaded part of figure 2 $\Phi^2_+$ and $\Phi^2_{\mathrm{CV}}$ evaluated at $\tilde{r}$ are both less than one, and obviously $\Phi^2_{\mathrm{CV}} > \Phi^2_+$ at $\tilde{r}$. The shape of these functions in this case is shown in figure 3(*b*). One can see that there is a family of $\theta =$ constant trajectories which enter their CV region before meeting a turning point (shaded area in figure 3(*b*)).

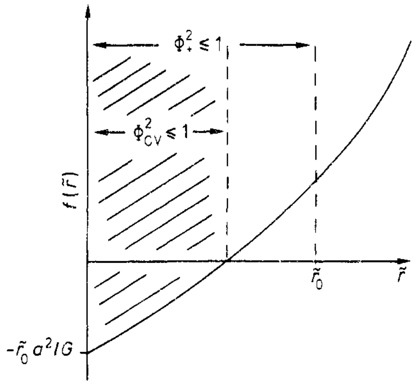


**Figure 2.** The function $f(\tilde r)$ is shown; in the shaded area, both $\Phi_+^2$ and $\Phi_{CV}^2$ evaluated at $\tilde r$ are less than one.

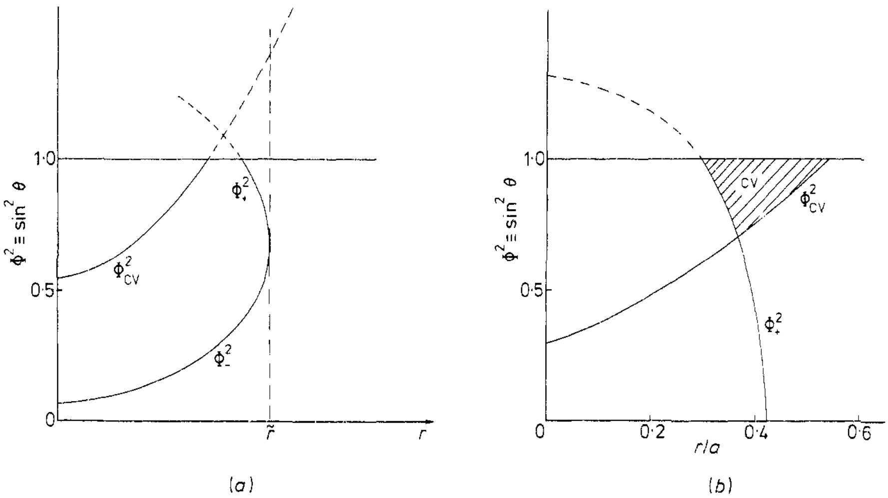


**Figure 3.** (*a*) The functions $\Phi_\mp^2$ (loci of inversion points) and $\Phi_{CV}^2$ are shown. In this case a particle moving on a $\theta$ = constant trajectory meets a turning point before entering its CV region. (*b*) This picture is drawn for $a = M = 1$, $\beta = -3$, $Q^2 = 5$, $G = \frac{5}{3}$. All the generalised radial trajectories entering the shaded area will meet a turning point after having entered the CV region. They can therefore make up for lost time and go back to infinity.

## 4. Conclusions

The general theory of relativity, although widely accepted as the most satisfactory theory for gravity, contains its own limits, as it predicts the occurrence of space–time singularities (see e.g. Hawking and Ellis 1973). The behaviour of the space–time in their vicinity is not yet completely understood; it is however remarkable that singularities predicted by some exact solutions, like Kerr and Kerr–Newman, lead to pathologies which seem to be unavoidable. This is the case for the causality violating paths which have been shown to exist (Carter 1968, Tipler 1976). No explicit examples however were known of time-like paths which violated causality in the sense stated earlier

(Calvani *et al* 1978, de Felice and Calvani 1979) and which connected the 'pathological' inner regions of the space–time with the asymptotic ones. In this paper we have a whole family of them. A similar example was found by considering null geodesics in the Kerr metric (de Felice and Calvani 1979); while in that case the corresponding CV conditions were met in the $r < 0$ part of the metric, which is of a dubious physical significance, here this effect takes place in some extended portion of the $r > 0$ part of the metric. This suggests that covering the singularity with a suitable material source may not be sufficient to take care of this effect.

The space–time singularities remain therefore the most challenging problem of today's relativity.

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